SVARC-MILNOR LEMMA: A PROOF BY DEFINITION

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Abstract. The famous Švarc-Milnor Lemma says that a group G acting properly and cocompactly via isometries on a length space X is finitely generated and induces a quasi-isometry equivalence $g \to g \cdot x_0$ for any $x_0 \in X$. We redefine the concept of coarseness so that the proof of the Lemma is automatic.

Geometric group theorists traditionally restrict their attention to finitely generated groups equipped with a word metric. A typical proof of Švarc-Milnor Lemma (see [5] or [1], p.140) involves such metrics. Recently, the study of large scale geometry of groups was expanded to all countable groups by usage of proper, left-invariant metrics: in [6] such metrics were constructed and it was shown that they all induce the same coarse structure on a group (see also [2]). The point of this note is that a proper action of a group G on a space ought to be viewed as a geometric way of creating a coarse structure on G. That structure is not given by a proper metric but by something very similar; a pseudo-metric where only a finite set of points may be at mutual distance 0. From that point of view the proof of Švarc-Milnor Lemma is automatic and the Lemma can be summarized as follows. There are two ways of creating coarse structures on countable groups: algebraic (via word or proper metrics) and geometric (via group actions), and both ways are equivalent.

Definition 0.1. A pseudo-metric d_X on a set X is called a *large-scale metric* (or ls-metric) if for each $x \in X$ the set $\{y \in X \mid d_X(x,y) = 0\}$ is finite.

(X, d_X) is called a *large-scale metric space* (or an ls-metric space) if d_X is an ls-metric.

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Definition 0.2. An ls-metric d_G on a group G is *proper* and *left-invariant* if $d_G(g,h) = d_G(f \cdot g, f \cdot h)$ for all $f,g,h \in G$ and $\{h \mid d_G(g,h) < r\}$ is finite for all $r > 0$ and all $g \in G$.

Notice G must be countable if it admits a proper ls-metric.

One aspect of Švarc-Milnor Lemma is G being finitely generated. That corresponds to (G, d_G) being metrically connected, i.e. there is $M > 0$ such that any two points in G can be connected by a chain of points separated by at most M.

Lemma 0.3. *Suppose d_G is a proper and left-invariant ls-metric on G. (G, d_G) is metrically connected if and only if G is finitely generated.*

Proof. If G is generated by a finite set F, put $M = \max\{d_G(1_G, f) \mid f \in F\}$. If (G, d_G) is M-connected, put $F = B(1_G, M+1)$. ■

Definition 0.4. A function $f\colon (X, d_X) \to (Y, d_Y)$ of ls-metric spaces is called *large-scale uniform* (or ls-uniform) if for each $r > 0$ there is $s > 0$ such that $d_X(x,y) \le r$ implies $d_Y(f(x), f(y)) \le s$.

f is a *large-scale uniform equivalence* if there is an ls-uniform $g : Y \to X$ such that both $g \circ f$ and $f \circ g$ are within a finite distance from the corresponding identities.

Lemma 0.5. *Suppose (G, d_G) and (H, d_H) are two groups equipped with proper and left-invariant ls-metrics. A function $f\colon (G, d_G) \to (H, d_H)$ is ls-uniform if and only if for each finite subset F of G there is a finite subset E of H such that $x^{-1} \cdot y \in F$ implies $f(x)^{-1} \cdot f(y) \in E$ for all $x, y \in G$.*

Proof. Suppose f is ls-uniform and F is a finite subset of G. Let r be larger that all $d_X(1_G, g)$, $g \in F$. Pick $s > 0$ such that $d_G(g,h) < r$ implies $d_H(f(g), f(h)) < s$ and put $E = \{x \in H \mid d_H(1_H, x) < s\}$. If $x^{-1} \cdot y \in F$, then $d_G(x,y) < r$. Therefore $s > d_H(f(x), f(y)) = d_H(1_H, f(x)^{-1} \cdot f(y))$ and $f(x)^{-1} \cdot f(y) \in E$. Conversely, if $r > 0$ put $F = \{x \in G \mid d_G(1_G, x) < r\}$ and consider E so that $x^{-1} \cdot y \in F$ implies $f(x)^{-1} \cdot f(y) \in E$. If s is bigger that all $d_H(1_H, g)$, $g \in E$, then $d_G(x,y) < r$ implies $f(x)^{-1} \cdot f(y) \in E$ and $d_H(f(x), f(y)) < s$. ■

Corollary 0.6. *Given two proper and left-invariant ls-metrics d_1 and d_2 on the same group G, the identity $id_G : (G, d_1) \to (G, d_2)$ is a coarse equivalence.*

Proof. The choice of $E = F$ always works for id_G. ■

We are interested in creating proper left-invariant ls-metrics on groups G using actions on metric spaces X via the formula $d_G(g,h) = d_X(g \cdot x_0, h \cdot x_0)$ for some $x_0 \in X$. To make d_G left-invariant, a practical

requirement is the action occurs via isometries. Let's characterize the situation in which d_G is a proper ls-metric.

Lemma 0.7. *Suppose G acts via isometries on X and $x_0 \in X$. If d_G is defined by $d_G(g,h) = d_X(g\cdot x_0, h\cdot x_0)$, then d_G is a proper left-invariant ls-metric on G if and only if the following conditions are satisfied:*

1. *The stabilizer $\{g \in G \mid g\cdot x_0 = x_0\}$ of x_0 is finite.*
2. *$G\cdot x_0$ is topologically discrete.*
3. *Every bounded subset of $G\cdot x_0$ that is metrically discrete is finite.*

Proof. Recall that A is metrically discrete if there is $s > 0$ such that $d_X(a,b) > s$ for all $a, b \in A$, $a \neq b$. Clearly, if one of Conditions 1-3 is not valid, then there is $r > 0$ such that $B(1_G, r)$ is infinite and d_G is not proper. Thus, assume 1-3 hold. Suppose $B(1_G, r)$ is infinite for some $r > 0$ and pick g_1 in that set. Suppose $\{g_n\}_{n=1}^k \subset B(1_G, 2r)$ is constructed so that $d_X(g_i\cdot x_0, x_0) < \frac{1}{i}$. Put $A = B(1_G, r) \setminus \{g_n\}_{n=1}^k$ and notice $A\cdot x_0$ is infinite (otherwise the stabilizer of x_0 is infinite). Hence there are two different elements $g, h \in A$ such that $g\cdot x_0 \neq h\cdot x_0$ and $d_X(g\cdot x_0, h\cdot x_0) < \frac{1}{k+1}$. Put $g_{k+1} = g^{-1}\cdot h$. However, $g_n\cdot x_0 \to x_0$, a contradiction. ■

It turns out, for nice spaces X, d_G being a proper ls-metric is equivalent to the action being proper.

Corollary 0.8. *Suppose (X, d_X) is a metric space so that all infinite bounded subsets of X contain an infinite Cauchy sequence. If a group G acts via isometries on X and $x_0 \in X$, then $d_G(g,h) = d_X(g\cdot x_0, h\cdot x_0)$ defines a proper left-invariant ls-metric on G if and only if there is a neighborhood U of x_0 such that the set $\{g \in G \mid g\cdot U \cap U \neq \emptyset\}$ is finite.*

Proof. Suppose there is a neighborhood U of x_0 such that the set $\{g \in G \mid g\cdot U \cap U \neq \emptyset\}$ is finite. Notice there is no converging sequence $g_n\cdot x_0 \to x_0$ with g_n's being all different.

If d_G is proper, then choose any ball $U = B(x_0, r)$ around x_0. Now, $g\cdot U \cap U \neq \emptyset$ means there is $x_g \in U$ so that $d_X(g\cdot x_g, x_0) < r$. Therefore $d_G(g, 1_G) = d_X(g\cdot x_0, x_0) \le d_X(g\cdot x_0, g\cdot x_g) + d_X(g\cdot x_g, x_g) + d_X(x_g, x_0) \le r + 2r + r = 4r$ and there are only finitely many such g's. ■

Corollary 0.9. *If a group G acts cocompactly and properly via isometries on a proper metric space X, then $g \to g\cdot x_0$ induces a coarse equivalence between G and X for all $x_0 \in X$.*

Proof. Define $d_G(g,h) = d_X(g\cdot x_0, h\cdot x_0)$ for all $g, h \in G$. Clearly, d_G is left-invariant. Since action is proper, d_G is a proper ls-metric. Since action is cocompact, X is within bounded distance from $G\cdot x_0$. ■

Corollary 0.10 (Švarc-Milnor)**.** *A group G acting properly and cocompactly via isometries on a length space X is finitely generated and induces a quasi-isometry equivalence $g \to g \cdot x_0$ for any $x_0 \in X$.*

Proof. Consider the proper left-invariant metric d_G induced on G by the action. The cocompactness of the action implies $G \cdot x_0$ is metrically connected. So is (G, d_G) and G must be finitely generated. Both X and a Cayley graph of G are proper geodesic spaces. Therefore any coarse equivalence between them is a quasi-isometric equivalence. ■

Final comments.

Let us point out that Švarc-Milnor Lemma 0.9 for non-finitely generated groups is useful when considering spaces of asymptotic dimension 0. A large scale analog $\mathcal{M}^0$ of 0-dimensional Cantor set is introduced in [3]: it is the set of all positive integers with ternary expression containing 0's and 2's only (with the metric from $\mathbb{R}_+$):

$$\mathcal{M}^0 = \{\sum_{i=-\infty}^{\infty} a_i 3^i \mid a_i = 0, 2\}.$$

Proposition 0.11. [3, Theorem 3.11] *The space $\mathcal{M}^0$ is universal for proper metric spaces of bounded geometry and of asymptotic dimension zero.*

Proposition 0.12. *The space $\mathcal{M}^0$ is coarsely equivalent to $\bigoplus\limits_{i=1}^{\infty} \mathbb{Z}_2$.*

Proof. Consider the subset $A = \{\sum\limits_{i=0}^{\infty} a_i 3^i \mid a_i = 0, 2\}$ of $\mathcal{M}^0$. Notice $\mathcal{M}^0$ is within bounded distance from A, so $A \to \mathcal{M}^0$ is a coarse equivalence. Also, there is an obvious action of $\bigoplus\limits_{i=1}^{\infty} \mathbb{Z}_2$ on A (flipping $a_i = 0$ to 2 or $a_i = 2$ to 0 if the corresponding term in $\bigoplus\limits_{i=1}^{\infty} \mathbb{Z}_2$ is not zero) that is proper and cocompact. ■

Notice any infinite countable group G of asymptotic dimension 0 is locally finite (see [6]). Thus it can be expressed as the union of a strictly increasing sequence of its finite subgroups $G_1 \subset G_2 \subset \ldots$ Put $n_1 = |G_1|$, $n_i = |G_i/G_{i-1}|$ for $i > 1$, and observe (using 0.5) that G is coarsely equivalent to $\bigoplus\limits_{i=1}^{\infty} \mathbb{Z}_{n_i}$. We do not know if any two infinite countable groups of asymptotic dimension 0 are coarsely equivalent.

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